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SOME COMMENTS ON AN APPROXIMATION BY AWOJOBI†

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Abstract—In several papers Awojobi has used an approximation technique to solve the problems of the forced oscillations of a body on an elastic stratum. The present paper compares his approximate solution for axially symmetric torsion with numerical results derived from an integral equation developed by Gladwell. The approximate results show acceptable accuracy for the prediction of resonant frequencies. The prediction of resonant amplitude is limited by the nature of the approximate method.

1. INTRODUCTION

Although there are presently many papers dealing with the vibrations of a body on an elastic half-space, there are relatively few which deal with the vibrations of a body on an elastic stratum. The reason is that the analysis is quite tedious, and the development of numerical results over a significantly wide frequency range requires a relatively large computing effort. At present exact solutions are available only for the case of axially symmetric torsion. The problem was first treated by Collins[1] who considered the forced torsional oscillations of an elastic half-space and an elastic stratum. He used a method developed previously to establish an integral equation which could be solved by iteration for low frequencies and relatively large stratum thicknesses. Williams[2] solved the same problem by using Green's function technique and also pointed out a minor error in Collins' paper. A simpler analysis was given by Gladwell[3] who formulated the problem by means of integral transform techniques. He obtained an integral equation which was solved asymptotically for the cases of low frequency and relatively large stratum thickness. Gladwell also considered the case when the bottom surface of the stratum is stress free.

The forms of solution as given by Collins, Williams and Gladwell provide a good vehicle for solution on a high speed computer, thereby yielding highly accurate numerical results over a relatively large frequency range. However, the difficulty of the computing problem is such that approximate solutions that could reduce the computational task appreciably would be highly desirable. It is for this purpose that Awojobi, in his papers[4–6], developed an approximation function to solve the respective dual integral equations. He deduced that if the problem of a body oscillating on an elastic stratum is formulated by integral transform techniques, then hyperbolic functions arise naturally in the resulting integral equations. Awojobi proposed to solve these equations by use of the approximation

$$\tanh(x) \doteq x/(1 + x^2)^{1/2} \tag{1}$$

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and the results in his papers followed from the substitution given by equation (1). The remainder of this paper will be concerned with numerical results derived from the solution of the integral equation in Gladwell's paper. It will be possible to obtain thereby the dimensionless amplitude at resonance, which can be compared with similar quantities from Awojobi's results.

2. SOLUTION

The problem of the axially symmetric torsional oscillations of a rigid disk of radius a, bonded to an elastic layer of thickness h, is considered. Through the use of integral transforms, Gladwell reduced the problem to a Fredholm integral equation of the second kind given by

$$\theta(x) + \frac{1}{\pi}\int_0^1 M^*(x, \xi)\theta(\xi)\,\mathrm{d}\xi = 2\phi x \qquad (0 \le x \le 1), \tag{2}$$

where $\theta(x)$ is an unknown function and the kernel is given as

$$\begin{aligned} M^*(x, \xi) = {} & k\int_0^1 \tan(\tau u)\Psi(k, \sqrt{1-u^2})\,\mathrm{d}u \\ & - k\int_0^1 \chi(k, \sqrt{1-u^2})\,\mathrm{d}u \\ & + k\int_0^\infty [\tanh(\tau u) - 1]\Psi(k, \sqrt{1+u^2})\,\mathrm{d}u \\ & - \frac{2\pi i}{d}\sum_{n=1}^{m} \sin(\xi_n x)\sin(\xi_n \xi). \end{aligned} \tag{3}$$

Here ϕ is the angle of twist at maximum amplitude, $d = h/a$, $k = a\omega(\rho/\mu)^{1/2}$, $\tau = kd$, μ and ρ are the shear modulus and density of the stratum and ω is the circular frequency. The functions χ and Ψ are defined by

$$\chi(k, s) = \sin[k|x - \xi|s] - \sin[k(x + \xi)s], \tag{4}$$

$$\Psi(k, s) = \cos[k(x - \xi)s] - \cos[k(x + \xi)s], \tag{5}$$

and ξ_n are the roots of the equation

$$\cos[(k^2 - \xi_n{}^2)^{1/2}\,d] = 0, \tag{6}$$

or ξ_n are given by

$$\xi_n = k\left\{1 - \left[\frac{(2n-1)\pi}{2\tau}\right]^2\right\}^{1/2} \quad (n = 1, 2, \ldots, m), \tag{7}$$

where m is the greatest integer less than $\tau/\pi + \frac{1}{2}$.

The computing task involves the computation of $\theta(x)$ from equation (2). One serious problem of the calculation is the evaluation of principal value integrals that occur in the first integral of equation (3). The number of principal values will depend upon the magnitude of the value of τ. Once $\theta(x)$ is obtained, the physical quantities are readily determined. Gladwell showed that the torque may be expressed as

$$T = 8\mu a^3 \int_0^1 t\theta(t)\,\mathrm{d}t. \tag{8}$$

Unless the value of τ is such that equation (6) has not roots, the torque will generally consist of a real and imaginary part and may be expressed as

$$T/(a\mu^3\phi) = \frac{16}{3}(1+p) + \frac{16}{3}q, \tag{9}$$

where p is the real part that deviates from the static half-space solution and q is the imaginary part of the torque. In terms of p and q the nondimensional amplitude may be given as

$$A = \left\{\left[\frac{16}{3}(1+p) - Jk^2\right]^2 + \left(\frac{16}{3}q\right)^2\right\}^{-1/2}, \tag{10}$$

where J is the nondimensional polar moment of inertia of the body and is related to the polar moment of inertia by the relation $J = J_p/\rho a^5$.

The amplitudes in equation (10) represent important physical quantities, and for comparison purposes the peak nondimensional amplitude as a function of d, for various values of J, would offer a fair test for the approximation scheme in[4].

Some comments should be made concerning the manner in which the integral equation (2) was solved. The principal value integration was computed by assuming the integrand to be anti-symmetric about its singular points over a sufficiently small interval. By disregarding this interval of 0·002, it was found that the integral could be computed to within 0·2 per cent accuracy.

Having developed a sufficiently accurate scheme for the calculation of the principal value integrals, integral equation (2) can be computed with relative ease. The integral was replaced by a finite sum by the use of Simpson's rule, and the integral equation was thereby transformed into a matrix equation which could be easily solved. The unknown function was solved by using eleven points each for its real and imaginary part. When 21 increments each were used the results were within 0·2 per cent of the previous case.

3. NUMERICAL RESULTS AND CONCLUSIONS

The complex stiffnesses were calculated for several values of d and a range of k from 0·8 to 1·5. The frequency, k, could be assigned higher values than the range given, but for the purposes of this paper a maximum value of 1·5 is sufficient. One interesting feature of the curves is that the real part changes its curvature when the frequency is sufficiently high as to cause participation of an additional root in equation (3). The imaginary part of the solution intersects the abscissa which further confirms the numerical procedures.

With the curves of complex stiffnesses obtained over a suitable frequency range it is possible to calculate the maximum amplitude for a given J and the frequency at which this amplitude occurs. The calculation is summarized by Table 1. The resonant frequencies calculated by the approximate method are in acceptable agreement with those of Table 1.

Figure 1 shows the curves drawn from Table 1 and points taken from equivalent curves in[4] and [5]. One should not expect qualitative agreement for large values of J and small values of d since the approximate technique predicts damping for all frequencies considered

Table 1. Resonance frequencies k_r and nondimensional amplitudes at resonance A_r

d	$J=5$		$J=3{\cdot}49$		$J=2$	
	k_r	A_r	k_r	A_r	k_r	A_r
1	1·011	∞	1·181	∞	1·459	∞
1·25	0·988	∞	1·145	∞	1·350	2·760
1·5	0·971	∞	1·098	6·031	1·341	1·185
1·75	0·944	8·499	1·088	2·121	1·355	0·924
2	0·939	3·114	1·095	1·592	1·379	0·843
2·5	0·948	2·069	1·113	1·363	1·426	0·847
3	0·957	1·946	1·127	1·392	1·454	0·966
3·5	0·963	2·008	1·135	1·510	1·405	1·058
4	0·959	2·366	1·134	1·726	1·400	0·951
4·5	0·967	2·363	1·118	1·680	1.404	0·913
5	0·959	2·462	1·118	1·549	1·429	0·931
10	0·961	2·178	1·124	1·596	1·417	0·958
∞	0·961	2·245	1·124	1·564	1·404	0·925

here. Equation (3) will have no complex part for the frequency range below a critical frequency which corresponds to the fundamental frequency of torsional vibration of a rod of the material of the stratum, of length h, free at one end and fixed at the other. Therefore below this frequency there will be no damping. The approximate method is seen to be reasonably effective for $d > 3\,(J=2)$, $d > 4\,(J=3{\cdot}49)$, and $d > 4{\cdot}5\,(J=5)$.

One can use the approximate method to predict resonant frequencies with reasonable accuracy. However, one must use caution in assigning damping with the approximate technique, since for certain ranges of parameter there will be qualitative differences between it and exact theory.

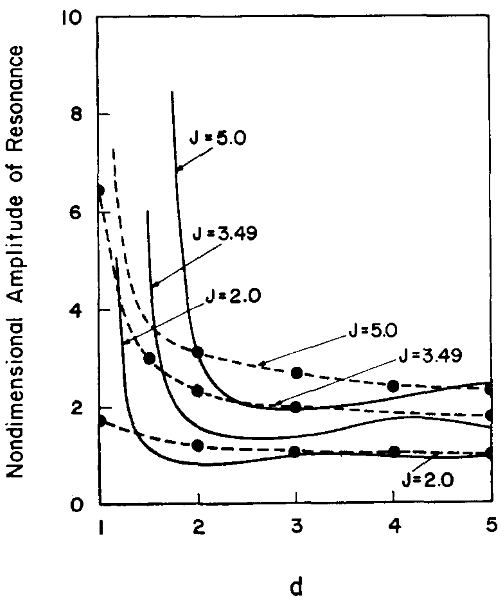

Fig. 1. Nondimensional amplitude at resonance as a function of d for $J = 2{\cdot}0$, $3{\cdot}49$ and $5{\cdot}0$. Circles show points taken from Figs. in [4] and [5]. Solid curves are from present analysis.

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Абстракт — В несколько работах Ауоджоби применял приближенный метод для расчета задач вымужденных колебаний тела на упругом слое. Настоящая работа сравнивает его приближенное решение, для осе-симметрического кручения, с численными результатами, полученными из интегрального уравнения, предложенного Гладуеллем. Приближенные результаты указывают желанную точность для предсказания резонансных частот. Предсказание резонансной амплитуды ограничено природой приближенного метода.